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Oscillation criteria for even order nonlinear neutral differential equations*

Yibing Sun[a]
[a] *School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China*
e-mail: sun_yibing@126.com
Zhenlai Han[a]
[a] *School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China*
e-mail: hanzhenlai@163.com
Shurong Sun[a,b]
[a] *School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China*
[b] *Department of Mathematics and Statistics,*
Missouri University of Science and Technology, Rolla, Missouri 65409-0020, USA
e-mail: sshrong@163.com
Chao Zhang[a]
[a] *School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, P R China*
e-mail: ss_zhangc@ujn.edu.cn

Abstract: In this paper, we consider the oscillation criteria for even order nonlinear neutral differential equations of the form

$$\left(r(t)z^{(n-1)}(t)\right)' + q(t)f(x(\sigma(t))) = 0,$$

where $z(t) = x(t) + p(t)x(\tau(t))$, $n \geq 2$ is a even integer. The results are obtained both for the case $\int^{\infty} r^{-1}(t)\mathrm{d}t = \infty$, and in case $\int^{\infty} r^{-1}(t)\mathrm{d}t < \infty$. These criteria here derived extend and improve some known results in literatures. Some examples are given to illustrate our main results.

Keywords: Oscillation; Even order; Nonlinear neutral differential equations
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1 Introduction

Over the last several years, there has been an increasing interest in the study of the oscillation theory and asymptotic behavior of solutions of differential equations. Recently, the applications of differential equations have been and still are receiving intensive attention and several monographs. There has been much research activity concerning the oscillatory behavior of the solutions of second order differential equations and second order neutral differential equations; see, for example, [1–18]. Up to now, many studies have been done on the oscillation problem of even order differential equations, and we refer the reader to the papers [19–29] and the references cited therein.

In this paper, we concerned with the oscillation theorems for the following even order half-linear neutral delay differential equation

$$\left(r(t)z^{(n-1)}(t)\right)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.1}$$

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where $z(t) = x(t) + p(t)x(\tau(t))$, $n \geq 2$ is a even integer. Throughout this paper, we assume that:

(C_1) $r \in C([t_0,\infty), R)$, $r(t) > 0$, $r'(t) \geq 0$;

(C_2) $p,\ q \in C([t_0,\infty), R)$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) > 0$, where p_0 is a constant;

(C_3) $\tau \in C^1([t_0,\infty), R)$, $\sigma \in C([t_0,\infty), R)$, $\tau'(t) \geq \tau_0 > 0$, $\sigma(t) \leq t$, $\tau\circ\sigma = \sigma\circ\tau$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$, where τ_0 is a constant;

(C_4) $f \in C(R,R)$ and $f(y)/y \geq L > 0$, for $y \neq 0$, L is a constant.

We shall also consider the two cases

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t = \infty, \tag{1.2}$$

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t < \infty. \tag{1.3}$$

By a solution x of (1.1) we mean a function $z \in C^{n-1}([t_x,\infty), R)$ for some $t_x \geq t_0$, where $z(t) = x(t) + a(t)x(\tau(t))$, which has the property that $rz^{(n-1)} \in C^1([t_x,\infty), R)$ and satisfies (1.1) on $[t_x,\infty)$. We consider only those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$. We assume that (1.1) possess such solutions. A nontrivial solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. (1.1) is said to be oscillatory if all its solutions are oscillatory.

For the particular case when $n = 2$, (1.1) reduces to the following equations

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0. \tag{1.4}$$

Han et al. [9] studied the oscillation criteria for the solutions of (1.4), where $\int_{t_0}^{\infty} r^{-1}(t)\mathrm{d}t = \infty$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $0 \leq p(t) \leq p_0 < \infty$.

In 2011, Baculíková and Džurina [13] studied the oscillatory behavior of the solutions of the second order neutral differential equations

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)x(\sigma(t)) = 0,\ t \geq t_0, \tag{1.5}$$

where $\int_{t_0}^{\infty} r^{-1}(s)\mathrm{d}s = \infty$, $0 \leq p(t) \leq p_0 < \infty$. Basing on the new comparison principles, the authors obtained some sufficient conditions for the oscillation of (1.5), which reduce the problem of the oscillation of the second order differential equations to the oscillation of a first order differential inequality. In this paper, Theorem 1 is quite general, since usual restrictions on the coefficients of (1.5), like $\tau(t) \leq t$, $\sigma(t) \leq \tau(t)$, $\sigma(t) \leq t$, $0 \leq p(t) < 1$, etc. are not assumed. Further, τ could be a delay or advanced argument, and σ could be a delay argument, hence the results obtained here improved and extended some known results in literature, such as [1, 5, 7].

Zhang et al. [26] studied the even-order nonlinear neutral functional differential equations

$$\big(x(t) + p(t)x(\tau(t))\big)^{(n)} + q(t)f\big(x(\sigma(t))\big) = 0,\ t \geq t_0, \tag{1.6}$$

where n is even, $0 \leq p(t) < 1$ and $\tau(t) \leq t$. The authors established a comparison theorem for (1.6) and the obtained results improved and generalized some known results. Using the Riccati transformation technique, Li et al. [25] obtained some new oscillation criteria for (1.6), when $0 \leq p(t) \leq p_0 < \infty$. These oscillation criteria, at least in some sense, complemented and improved those of Zafer [20] and Zhang et al. [26].

In 2011, Zhang et al. [28] studied the oscillatory behavior of the following higher-order half-linear delay differential equation

$$\Big(r(t)(x^{(n-1)}(t))^{\alpha}\Big)' + q(t)x^{\beta}(\tau(t)) = 0,\quad t \geq t_0, \tag{1.7}$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\mathrm{d}t < \infty.$$

The authors obtained some sufficient conditions, which guarantee that every solution of (1.7) is oscillatory or tends to zero.

Clearly, the above equations are special cases of (1.1). To the best of our knowledge, there are few results regarding the oscillation criteria for (1.1) under the condition (1.3). The purpose of this paper is to derive some oscillation theorems of (1.1). Our results obtained here improve and extend the main results of [9–11, 13, 20, 23, 25, 26].

2 Some preliminary lemmas

In this section, we present some useful lemmas, which will be used in the proofs of our main results.

Lemma 2.1 *[29] Let $u \in C^n([t_0, \infty), R^+)$. If $u^{(n)}(t)$ is eventually of one sign for all large t, then there exist a $t_x > t_1$, for some $t_1 > t_0$, and an integer l, $0 \le l \le n$, with $n + l$ even for $u^{(n)}(t) \ge 0$ or $n + l$ odd for $u^{(n)}(t) \le 0$ such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t > t_x$, $k = 0,\ 1, ...,\ l - 1$, and $l \le n - 1$, implies that $(-1)^{l+k}u^{(k)}(t) > 0$ for $t > t_x$, $k = l,\ l + 1, ...,\ n - 1$.*

Lemma 2.2 *[19] Let u be as in Lemma 2.1. Assume that $u^{(n)}(t)$ is not identically zero on any interval $[t_0, \infty)$, and there exists a $t_1 \ge t_0$ such that $u^{(n-1)}(t)u^{(n)}(t) \le 0$ for all $t \ge t_1$. If $\lim_{t\to\infty} u(t) \ne 0$, then for every λ, $0 < \lambda < 1$, there exists $T \ge t_1$, such that for all $t \ge T$,*

$$u(t) \ge \frac{\lambda}{(n-1)!} t^{n-1} u^{(n-1)}(t).$$

Lemma 2.3 *Assume that (1.2) holds. Furthermore, assume that x is an eventually positive solution of (1.1). Then there exists $t_1 \ge t_0$, such that*

$$z(t) > 0, \quad z'(t) > 0, \quad z^{(n-1)}(t) > 0 \quad \text{and} \quad z^{(n)}(t) \le 0, \quad \text{for all} \quad t \ge t_1.$$

The proof is similar to that of Meng and Xu [24, Lemma 2.3], so is omitted.

Lemma 2.4 *[18, Theorem 2.1.1] Consider the oscillatory behavior of solutions of the following linear differential inequality*

$$y'(t) + p(t)y(\tau(t)) \le 0, \tag{2.1}$$

where $p,\ \tau \in C([t_0, \infty), (0, \infty))$, $\tau(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$. If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\mathrm{d}s > \frac{1}{e},$$

then (2.1) has no eventually positive solutions.

3 Main results

In this section, we state the main results which guarantee that every solution of (1.1) is oscillatory.

Theorem 3.1 *Assume that (1.2) holds. If*

$$\int_{t_0}^{\infty} P(t)\mathrm{d}t = \infty, \tag{3.1}$$

where $P(t) = \min\{q(t),\ q(\tau(t))\}$, then every solution of (1.1) is oscillatory.

Proof. Suppose, on the contrary, x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \ge t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \ge t_1$. Using the definition of z and Lemma 2.3, we have $z(t) > 0$, $z'(t) > 0$, $z^{(n-1)}(t) > 0$ and $z^{(n)}(t) \le 0$, $t \ge t_1$. Hence, $\lim_{t\to\infty} z(t) \ne 0$. Applying (C_4) and (1.1), we get

$$\left(r(t)z^{(n-1)}(t)\right)' \le -Lq(t)x(\sigma(t)) < 0, \quad t \ge t_1.$$

Therefore, $r(t)z^{(n-1)}(t)$ is a decreasing function. Furthermore, from the above inequality and the definition of z, we obtain

$$\left(r(t)z^{(n-1)}(t)\right)' + Lq(t)x(\sigma(t)) + \frac{p_0}{\tau'(t)}\left(r(\tau(t))z^{(n-1)}(\tau(t))\right)' + Lp_0q(\tau(t))x(\sigma(\tau(t))) \le 0,$$

thus

$$\left(r(t)z^{(n-1)}(t)\right)' + LP(t)z(\sigma(t)) + \frac{p_0}{\tau_0}\left(r(\tau(t))z^{(n-1)}(\tau(t))\right)' \leq 0, \tag{3.2}$$

where P is defined as in Theorem 3.1. Integrating (3.2) from t_1 to t, we have

$$\int_{t_1}^{t}\left(r(s)z^{(n-1)}(s)\right)'\mathrm{d}s + L\int_{t_1}^{t}P(s)z(\sigma(s))\mathrm{d}s + \frac{p_0}{\tau_0}\int_{t_1}^{t}\left(r(\tau(s))z^{(n-1)}(\tau(s))\right)'\mathrm{d}s \leq 0.$$

Noticing that $\tau'(t) \geq \tau_0 > 0$, we get

$$\begin{aligned} L\int_{t_1}^{t}P(s)z(\sigma(s))\mathrm{d}s &\leq -\int_{t_1}^{t}\left(r(s)z^{(n-1)}(s)\right)'\mathrm{d}s - \frac{p_0}{\tau_0}\int_{t_1}^{t}\frac{1}{\tau'(s)}\left(r(\tau(s))z^{(n-1)}(\tau(s))\right)'\mathrm{d}(\tau(s)) \\ &\leq r(t_1)z^{(n-1)}(t_1) - r(t)z^{(n-1)}(t) \\ &\quad + \frac{p_0}{\tau_0^2}\left(r(\tau(t_1))z^{(n-1)}(\tau(t_1) - r(\tau(t))z^{(n-1)}(\tau(t))\right). \end{aligned} \tag{3.3}$$

Since $z'(t) > 0$ for $t \geq t_1$, we can find a constant $c > 0$ such that $z(\sigma(t)) \geq c$, $t \geq t_1$. Then from (3.3) and the fact that $r(t)z^{(n-1)}(t)$ is decreasing, we obtain

$$\int_{t_1}^{\infty}P(t)\mathrm{d}t < \infty,$$

which is in contradiction with (3.1). This completes the proof.

Remark 3.1 *Recently, when studying the properties of the neutral differential equations, there are many further restrictions on the coefficients, such as $\tau(t) \leq t$, $\sigma(t) \leq \tau(t)$, $0 \leq p(t) < 1$, etc. In Theorem 3.1 no such constraints are assumed, and therefore our results are of high generality.*

Theorem 3.2 *Assume that* (1.2) *holds and $\tau(t) \geq t$. If either*

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{(p_0+\tau_0)(n-1)!}{\tau_0 e}, \tag{3.4}$$

or when σ is nondecreasing,

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{(p_0+\tau_0)(n-1)!}{\tau_0}, \tag{3.5}$$

where $Q(t) = \min\{Lq(t),\ Lq(\tau(t))\}$, then every solution of (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we have (3.2). By Lemma 2.2 and (3.2), for every λ, $0 < \lambda < 1$, we obtain

$$\left(r(t)z^{(n-1)}(t)\right)' + \frac{p_0}{\tau_0}\left(r(\tau(t))z^{(n-1)}(\tau(t))\right)' + \frac{\lambda}{(n-1)!}\sigma^{n-1}(t)Q(t)z^{(n-1)}(\sigma(t)) \leq 0,$$

for every t sufficiently large. Let $u(t) = r(t)z^{(n-1)}(t) > 0$. Then for all t large enough, we have

$$\left(u(t) + \frac{p_0}{\tau_0}u(\tau(t))\right)' + \frac{\lambda}{(n-1)!}\frac{\sigma^{n-1}(t)Q(t)}{r(\sigma(t))}u(\sigma(t)) \leq 0. \tag{3.6}$$

Next, let us denote $\omega(t) = u(t) + \frac{p_0}{\tau_0}u(\tau(t))$. Since u is decreasing, it follows from $\tau(t) \geq t$ that

$$\omega(t) \leq \left(1 + \frac{p_0}{\tau_0}\right)u(t). \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\omega'(t) + \frac{\tau_0}{p_0+\tau_0}\frac{\lambda}{(n-1)!}\frac{\sigma^{n-1}(t)Q(t)}{r(\sigma(t))}\omega(\sigma(t)) \le 0. \tag{3.8}$$

Therefore, ω is a positive solution of (3.8). Now, we consider the following two cases, depending on whether (3.4) or (3.5) holds.

Case (I): It is easy to see that if (3.4) holds, then we can choose a constant $0 < \lambda_0 < 1$, such that

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t}\frac{\tau_0}{p_0+\tau_0}\frac{\lambda_0}{(n-1)!}\frac{\sigma^{n-1}(t)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{1}{e}. \tag{3.9}$$

But according to Lemma 2.4, (3.9) guarantees that (3.8) has no positive solution, which is a contradiction.

Case (II): Using the definition of ω and (3.2), we obtain

$$\omega'(t) = u'(t) + \frac{p_0}{\tau_0}(u(\tau(t)))' \le -Q(t)z(\sigma(t)) < 0. \tag{3.10}$$

Noting that $\sigma(t) \le t$, there exists $t_2 \ge t_1$, such that

$$\omega(\sigma(t)) \ge \omega(t), \quad t \ge t_2. \tag{3.11}$$

Integrating (3.8) from $\sigma(t)$ to t and applying σ is nondecreasing, we have

$$\omega(t) - \omega(\sigma(t)) + \frac{\tau_0}{p_0+\tau_0}\frac{\lambda}{(n-1)!}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\omega(\sigma(s))\mathrm{d}s \le 0, \quad t \ge t_2.$$

Thus

$$\omega(t) - \omega(\sigma(t)) + \frac{\tau_0}{p_0+\tau_0}\frac{\lambda}{(n-1)!}\omega(\sigma(t))\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s \le 0, \quad t \ge t_2.$$

From the above inequality, we obtain

$$\frac{\omega(t)}{\omega(\sigma(t))} - 1 + \frac{\tau_0}{p_0+\tau_0}\frac{\lambda}{(n-1)!}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s \le 0.$$

Hence from (3.11), we have

$$\frac{\tau_0}{p_0+\tau_0}\frac{\lambda}{(n-1)!}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s \le 1, \quad t \ge t_2. \tag{3.12}$$

Taking the upper limit as $t \to \infty$ in (3.12), we get

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s \le \frac{(p_0+\tau_0)(n-1)!}{\lambda\tau_0}. \tag{3.13}$$

If (3.5) holds, we can choose a constant $0 < \lambda_0 < 1$, such that

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{(p_0+\tau_0)(n-1)!}{\lambda_0\tau_0},$$

which is in contradiction with (3.13). This completes the proof.

Theorem 3.3 *Assume that* (1.2) *holds and* $\sigma(t) \le \tau(t) \le t$. *If either*

$$\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{(p_0+\tau_0)(n-1)!}{\tau_0 e}, \tag{3.14}$$

or when $\tau^{-1}\circ\sigma$ *is nondecreasing,*

$$\limsup_{t\to\infty}\int_{\tau^{-1}(\sigma(t))}^{t}\frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))}\mathrm{d}s > \frac{(p_0+\tau_0)(n-1)!}{\tau_0}, \tag{3.15}$$

where Q *is defined as in Theorem 3.2, then every solution of* (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.2, we have (3.6). Let $\omega(t) = u(t) + \frac{p_0}{\tau_0}u(\tau(t))$ again. Since u is decreasing, it follows from $\tau(t) \leq t$ that

$$\omega(t) \leq \left(1 + \frac{p_0}{\tau_0}\right) u(\tau(t)). \tag{3.16}$$

Combining (3.6) and (3.16), we get

$$\omega'(t) + \frac{\tau_0}{p_0 + \tau_0} \frac{\lambda}{(n-1)!} \frac{\sigma^{n-1}(t)Q(t)}{r(\sigma(t))} \omega(\tau^{-1}(\sigma(t))) \leq 0. \tag{3.17}$$

Therefore, ω is a positive solution of (3.17). Now, we consider the following two cases, depending on whether (3.14) or (3.15) holds.

Case (I): The proof is similar to the proof of Case (I) in Theorem 3.2, so it can be omitted.

Case (II): From (3.10) and the condition $\sigma(t) \leq \tau(t)$, there exists $t_2 \geq t_1$, such that

$$\omega(\tau^{-1}(\sigma(t))) \geq \omega(t), \quad t \geq t_2. \tag{3.18}$$

Integrating (3.17) from $\tau^{-1}(\sigma(t))$ to t and applying $\tau^{-1} \circ \sigma$ is nondecreasing, we get

$$\omega(t) - \omega(\tau^{-1}(\sigma(t))) + \frac{\tau_0}{p_0 + \tau_0} \frac{\lambda}{(n-1)!} \int_{\tau^{-1}(\sigma(t))}^{t} \frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))} \omega(\tau^{-1}(\sigma(s)))\mathrm{d}s \leq 0, \quad t \geq t_2.$$

Thus

$$\omega(t) - \omega(\tau^{-1}(\sigma(t))) + \frac{\tau_0}{p_0 + \tau_0} \frac{\lambda}{(n-1)!} \omega(\tau^{-1}(\sigma(t))) \int_{\tau^{-1}(\sigma(t))}^{t} \frac{\sigma^{n-1}(s)Q(s)}{r(\sigma(s))} \mathrm{d}s \leq 0, \quad t \geq t_2.$$

The rest of the proof is similar to that of Theorem 3.2, leading to a contradiction to (3.15), so it can be omitted. This completes the proof.

Theorem 3.4 *Assume that* (1.3) *holds and* $\sigma(t) \leq \tau(t) \leq t$. *If either* (3.14) *holds or when* $\tau^{-1} \circ \sigma$ *is nondecreasing,* (3.15) *holds and for sufficiently large* $t_1 \geq t_0$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\frac{\lambda_0}{(n-2)!} \delta(s)Q(s)\sigma^{n-2}(s) - \frac{1 + p_0/\tau_0}{4} \frac{1}{r(s)\delta(s)} \right] \mathrm{d}s = \infty, \tag{3.19}$$

where Q *is defined as in Theorem 3.2,* $0 < \lambda_0 < 1$ *is a constant and* $\delta(t) = \int_t^\infty r^{-1}(s)\mathrm{d}s$, *then every solution of* (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we can see that $r(t)z^{(n-1)}(t)$ is a decreasing function. Consequently it is easy to conclude that there exist two possible cases of the sign of $z^{(n-1)}(t)$, that is, $z^{(n-1)}(t)$ is either eventually positive or eventually negative for $t \geq t_2 \geq t_1$.

Case (I): $z^{(n-1)}(t) > 0$, $t \geq t_2$. The proof of this case is similar to that of Theorem 3.3, so we omit the details.

Case (II): $z^{(n-1)}(t) < 0$, $t \geq t_2$. Applying Lemma 2.1, we get $z^{(n-2)}(t) > 0$ and $z'(t) > 0$, then $\lim_{t\to\infty} z(t) \neq 0$. Define the function ω by

$$\omega(t) = \frac{r(t)z^{(n-1)}(t)}{z^{(n-2)}(t)}, \quad t \geq t_2. \tag{3.20}$$

Clearly, $\omega(t) < 0$ for $t \geq t_2$. Noting that $r(t)z^{(n-1)}(t)$ is decreasing, we obtain

$$r(s)z^{(n-1)}(s) \leq r(t)z^{(n-1)}(t), \quad s \geq t \geq t_2. \tag{3.21}$$

Dividing (3.21) by $r(s)$ and integrating it from t to l $(l \geq t)$, we have

$$z^{(n-2)}(l) \leq z^{(n-2)}(t) + r(t)z^{(n-1)}(t) \int_t^l \frac{1}{r(s)} \mathrm{d}s.$$

Letting $l \to \infty$, we get

$$0 \leq z^{(n-2)}(t) + r(t)z^{(n-1)}(t)\delta(t),$$

that is

$$-1 \leq \frac{r(t)z^{(n-1)}(t)}{z^{(n-2)}(t)}\delta(t),$$

where $\delta(t) = \int_t^\infty r^{-1}(s)\mathrm{d}s$. Therefore, from (3.20), we obtain

$$-1 \leq \omega(t)\delta(t) \leq 0, \quad t \geq t_2. \tag{3.22}$$

Similarly, we introduce a Riccati transformation

$$\nu(t) = \frac{r(\tau(t))z^{(n-1)}(\tau(t))}{z^{(n-2)}(t)}, \quad t \geq t_2. \tag{3.23}$$

Clearly, $\nu(t) < 0$ for $t \geq t_2$. Noting that $r(t)z^{(n-1)}(t)$ is decreasing and $\tau(t) \leq t$, we have $r(\tau(t))z^{(n-1)}(\tau(t)) \geq r(t)z^{(n-1)}(t)$, then $\nu(t) \geq \omega(t)$. Thus, by (3.22), we get

$$-1 \leq \nu(t)\delta(t) \leq 0, \quad t \geq t_2. \tag{3.24}$$

Differentiating (3.20), we obtain

$$\begin{aligned} \omega'(t) &= \frac{(r(t)z^{(n-1)}(t))'}{z^{(n-2)}(t)} - \frac{r(t)(z^{(n-1)}(t))^2}{(z^{(n-2)}(t))^2} \\ &= \frac{(r(t)z^{(n-1)}(t))'}{z^{(n-2)}(t)} - \frac{\omega^2(t)}{r(t)}. \end{aligned} \tag{3.25}$$

Differentiating (3.23) and from (3.21), we have

$$\begin{aligned} \nu'(t) &= \frac{(r(\tau(t))z^{(n-1)}(\tau(t)))'}{z^{(n-2)}(t)} - \frac{r(\tau(t))z^{(n-1)}(\tau(t))z^{(n-1)}(t)}{(z^{(n-2)}(t))^2} \\ &\leq \frac{(r(\tau(t))z^{(n-1)}(\tau(t)))'}{z^{(n-2)}(t)} - \frac{\nu^2(t)}{r(t)}. \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we get

$$\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq \frac{(r(t)z^{(n-1)}(t))'}{z^{(n-2)}(t)} + \frac{p_0}{\tau_0}\frac{(r(\tau(t))z^{(n-1)}(\tau(t)))'}{z^{(n-2)}(t)} - \frac{\omega^2(t)}{r(t)} - \frac{p_0}{\tau_0}\frac{\nu^2(t)}{r(t)}. \tag{3.27}$$

Therefore, by (3.2) and (3.27), we obtain

$$\omega'(t) + \frac{p_0}{\tau_0}\nu'(t) \leq -Q(t)\frac{z(\sigma(t))}{z^{(n-2)}(t)} - \frac{\omega^2(t)}{r(t)} - \frac{p_0}{\tau_0}\frac{\nu^2(t)}{r(t)}. \tag{3.28}$$

On the other hand, from Lemma 2.2, for every $0 < \lambda < 1$, we have

$$z(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}z^{(n-2)}(t). \tag{3.29}$$

Since $z^{(n-1)}(t) < 0$ and $\sigma(t) \leq t$, then

$$z^{(n-2)}(t) \leq z^{(n-2)}(\sigma(t)). \tag{3.30}$$

Thus, combining (3.28)–(3.30), we get

$$\omega'(t)+\frac{p_0}{\tau_0}\nu'(t)\le -\frac{\lambda}{(n-2)!}Q(t)\sigma^{n-2}(t)-\frac{\omega^2(t)}{r(t)}-\frac{p_0}{\tau_0}\frac{\nu^2(t)}{r(t)}. \tag{3.31}$$

Multiplying (3.31) by $\delta(t)$ and integrating from t_2 to t, we obtain

$$\begin{aligned}\delta(t)\omega(t)-\delta(t_2)\omega(t_2)&+\int_{t_2}^t\frac{\omega(s)}{r(s)}\mathrm{d}s+\int_{t_2}^t\frac{\omega^2(s)\delta(s)}{r(s)}\mathrm{d}s+\frac{p_0}{\tau_0}\delta(t)\nu(t)-\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2)\\ &+\frac{p_0}{\tau_0}\int_{t_2}^t\frac{\nu(s)}{r(s)}\mathrm{d}s+\frac{p_0}{\tau_0}\int_{t_2}^t\frac{\nu^2(s)\delta(s)}{r(s)}\mathrm{d}s+\frac{\lambda}{(n-2)!}\int_{t_2}^t\delta(s)Q(s)\sigma^{n-2}(s)\mathrm{d}s\le 0.\end{aligned} \tag{3.32}$$

It follows from (3.32), taking into account that $-1\le\omega(t)\delta(t)\le 0$, $-1\le\nu(t)\delta(t)\le 0$,

$$\begin{aligned}&\delta(t)\omega(t)-\delta(t_2)\omega(t_2)+\frac{p_0}{\tau_0}\delta(t)\nu(t)-\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2)\\ &+\frac{\lambda}{(n-2)!}\int_{t_2}^t\delta(s)Q(s)\sigma^{n-2}(s)\mathrm{d}s-\frac{1+p_0/\tau_0}{4}\int_{t_2}^t\frac{1}{r(s)\delta(s)}\mathrm{d}s\le 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\delta(t)\omega(t)+\frac{p_0}{\tau_0}\delta(t)\nu(t)+\int_{t_2}^t\left[\frac{\lambda}{(n-2)!}\delta(s)Q(s)\sigma^{n-2}(s)-\frac{1+p_0/\tau_0}{4}\frac{1}{r(s)\delta(s)}\right]\mathrm{d}s\\ \le\delta(t_2)\omega(t_2)+\frac{p_0}{\tau_0}\delta(t_2)\nu(t_2).\end{aligned}$$

From (3.19) and the above inequality, we get a contradiction to (3.22) and (3.24). This completes the proof.

Remark 3.2 *If $n=2$, the condition* (3.19) *of Theorem 3.4 becomes (3.2) of Theorem 3.1 in [9].*

Theorem 3.5 *Assume that* (1.3) *holds and $\tau(t)\ge t$. If either* (3.4) *holds or when σ is non-decreasing,* (3.5) *holds and for sufficiently large $t_1\ge t_0$,*

$$\limsup_{t\to\infty}\int_{t_1}^t\left[\frac{\lambda_0}{(n-2)!}\delta(\tau(s))Q(s)\sigma^{n-2}(s)-\frac{1+p_0/\tau_0}{4}\frac{(\tau'(s))^2}{r(s)\delta(\tau(s))}\right]\mathrm{d}s=\infty, \tag{3.33}$$

where Q is defined as in Theorem 3.2, $0<\lambda_0<1$ is a constant and δ is defined as in Theorem 3.4, then every solution of (1.1) *is oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a constant $t_1\ge t_0$, such that $x(t)>0$, $x(\tau(t))>0$ and $x(\sigma(t))>0$ for all $t\ge t_1$. Proceeding as in the proof of Theorem 3.1, we can see that $r(t)z^{(n-1)}(t)$ is a decreasing function. Consequently it is easy to conclude that there exist two possible cases of the sign of $z^{(n-1)}(t)$, that is, $z^{(n-1)}(t)$ is either eventually positive or eventually negative for $t\ge t_2\ge t_1$.

Case (I): $z^{(n-1)}(t)>0$, $t\ge t_2$. The proof of this case is similar to that of Theorem 3.2, so we omit the details.

Case (II): $z^{(n-1)}(t)<0$, $t\ge t_2$. Applying Lemma 2.1, we get $z^{(n-2)}(t)>0$ and $z'(t)>0$, then $\lim_{t\to\infty}z(t)\ne 0$. Define the function ν as (3.23). Since $r(t)z^{(n-1)}(t)$ is decreasing, we have

$$r(\tau(s))z^{(n-1)}(\tau(s))\le r(\tau(t))z^{(n-1)}(\tau(t)),\quad s\ge t\ge t_2. \tag{3.34}$$

Dividing (3.34) by $r(\tau(s))$ and integrating it from t to l ($l\ge t$), we get

$$z^{(n-2)}(\tau(l))\le z^{(n-2)}(\tau(t))+r(\tau(t))z^{(n-1)}(\tau(t))\int_{\tau(t)}^{\tau(l)}\frac{1}{r(s)}\mathrm{d}s.$$

Letting $l \to \infty$ in the above inequality, we obtain

$$0 \leq z^{(n-2)}(\tau(t)) + r(\tau(t))z^{(n-1)}(\tau(t))\delta(\tau(t)).$$

Noting that $z^{(n-1)}(t) < 0$ and $\tau(t) \geq t$, we have

$$z^{(n-2)}(\tau(t)) \leq z^{(n-2)}(t), \quad t \geq t_2.$$

Therefore,

$$-1 \leq \frac{r(\tau(t))z^{(n-1)}(\tau(t))}{z^{(n-2)}(t)}\delta(\tau(t)),$$

that is,

$$-1 \leq \nu(t)\delta(\tau(t)) \leq 0, \quad t \geq t_2, \tag{3.35}$$

where δ is defined as in Theorem 3.4. Next, define the function ω as (3.20). Noting that $r(t)z^{(n-1)}(t)$ is decreasing and $\tau(t) \geq t$, we get $r(\tau(t))z^{(n-1)}(\tau(t)) \leq r(t)z^{(n-1)}(t)$, $\omega(t) \geq \nu(t)$. Thus, by (3.35), we obtain

$$-1 \leq \omega(t)\delta(\tau(t)) \leq 0, \quad t \geq t_2. \tag{3.36}$$

We proceed as in the proof of Theorem 3.4 to get (3.31). Multiplying (3.31) by $\delta(\tau(t))$ and integrating from t_2 to t, we have

$$\delta(\tau(t))\omega(t) - \delta(\tau(t_2))\omega(t_2) + \int_{t_2}^{t} \frac{\omega(s)\tau'(s)}{r(s)}\mathrm{d}s + \int_{t_2}^{t} \frac{\omega^2(s)\delta(\tau(s))}{r(s)}\mathrm{d}s + \frac{p_0}{\tau_0}\delta(\tau(t))\nu(t) - \frac{p_0}{\tau_0}\delta(\tau(t_2))\nu(t_2)$$

$$+\frac{p_0}{\tau_0}\int_{t_2}^{t} \frac{\nu(s)\tau'(s)}{r(s)}\mathrm{d}s + \frac{p_0}{\tau_0}\int_{t_2}^{t} \frac{\nu^2(s)\delta(\tau(s))}{r(s)}\mathrm{d}s + \frac{\lambda}{(n-2)!}\int_{t_2}^{t} \delta(\tau(s))Q(s)\sigma^{n-2}(s)\mathrm{d}s \leq 0. \tag{3.37}$$

It follows from (3.37) that

$$\delta(\tau(t))\omega(t) - \delta(\tau(t_2))\omega(t_2) + \frac{p_0}{\tau_0}\delta(\tau(t))\nu(t) - \frac{p_0}{\tau_0}\delta(\tau(t_2))\nu(t_2)$$

$$+\frac{\lambda}{(n-2)!}\int_{t_2}^{t} \delta(\tau(s))Q(s)\sigma^{n-2}(s)\mathrm{d}s - \frac{1+p_0/\tau_0}{4}\int_{t_2}^{t} \frac{(\tau'(s))^2}{r(s)\delta(\tau(s))}\mathrm{d}s \leq 0.$$

Therefore,

$$\delta(\tau(t))\omega(t) + \frac{p_0}{\tau_0}\delta(\tau(t))\nu(t) + \int_{t_2}^{t} \left[\frac{\lambda}{(n-2)!}\delta(\tau(s))Q(s)\sigma^{n-2}(s) - \frac{1+p_0/\tau_0}{4}\frac{(\tau'(s))^2}{r(s)\delta(\tau(s))}\right]\mathrm{d}s$$

$$\leq \delta(\tau(t_2))\omega(t_2) + \frac{p_0}{\tau_0}\delta(\tau(t_2))\nu(t_2).$$

From (3.33) and the above inequality, we get a contradiction to (3.35) and (3.36). This completes the proof.

Remark 3.3 *The oscillation criteria from [9–11, 25] require condition $\tau(t) \leq t$, so they fail when $\tau(t) \geq t$. On the other hand, the oscillation criteria from [14, 20, 26] need $0 \leq p(t) < 1$, so they cannot be applied when $p(t) > 1$. Therefore, our results obtained here improve and complement those results.*

4 Examples

In this section, we will show the application of our main results.

Example 4.1 Consider the even order nonlinear neutral differential equations

$$\left(t^{\frac{1}{2}}(x(t) + p_0x(\alpha t))^{(n-1)}\right)' + \frac{a}{t^{n-\frac{1}{2}}}x(\beta t) = 0, \quad t \geq t_0. \tag{4.1}$$

Here $r(t) = t^{1/2}$, $\tau(t) = \alpha t$, $q(t) = a/t^{n-\frac{1}{2}}$, $\sigma(t) = \beta t$, $p(t) = p_0$, $0 < p_0 < \infty$, $f(x) = x$, $0 < \alpha < \infty$, $0 < \beta < 1$ and $a > 0$.

If $\alpha \geq 1$, then $Q(t) = q(\tau(t)) = a/(\alpha t)^{n-\frac{1}{2}}$ and conditions (3.4) or (3.5) of Theorem 3.2 reduces to

$$a\left(\frac{\beta}{\alpha}\right)^{n-\frac{3}{2}} \ln\frac{1}{\beta} > \frac{(\alpha+p_0)(n-1)!}{e} \tag{4.2}$$

or

$$a\left(\frac{\beta}{\alpha}\right)^{n-\frac{3}{2}} \ln\frac{1}{\beta} > (\alpha+p_0)(n-1)!,$$

respectively, which guarantees that every solution of (4.1) is oscillatory.

On the other hand, if $0 < \beta \leq \alpha \leq 1$, then $Q(t) = q(t) = a/t^{n-\frac{1}{2}}$ and conditions (3.14) or (3.15) of Theorem 3.3 reduces to

$$a\beta^{n-\frac{3}{2}} \ln\frac{\alpha}{\beta} > \frac{(\alpha+p_0)(n-1)!}{\alpha e} \tag{4.3}$$

or

$$a\beta^{n-\frac{3}{2}} \ln\frac{\alpha}{\beta} > \frac{(\alpha+p_0)(n-1)!}{\alpha},$$

respectively, which guarantees that every solution of (4.1) is oscillatory. Consequently, for all $\alpha > 0$, we cover the oscillation criteria for (4.1) whether $\tau(t) = \alpha t$ is delay or advanced argument. When $n = 2$, (4.1) becomes (E_5) in [13], and the conditions (4.2) and (4.3) reduce to the inequalities in Example 1 in [13]. So our results contain the main results in [13].

Example 4.2 Consider the even order nonlinear neutral differential equations

$$\left(t^\theta(x(t) + p_0 x(\alpha t))^{(n-1)}\right)' + (n-1)!t^{\theta-n}x(\beta t) = 0, \quad t \geq t_0 = 1. \tag{4.4}$$

Let $r(t) = t^\theta$, $\tau(t) = \alpha t$, $q(t) = (n-1)!t^{\theta-n}$, $\sigma(t) = \beta t$, $\theta \geq n$, $p(t) = p_0$, $0 < p_0 < \infty$, $f(x) = x$, $0 < \alpha < \infty$ and $0 < \beta < 1$.

If $\alpha \geq 1$, then $Q(t) = q(t) = (n-1)!t^{\theta-n}$. When

$$\beta^{n-\theta} \ln\frac{1}{\beta} > p_0 + \alpha,$$

it follows that (3.4) or (3.5) holds, respectively. Furthermore, from Theorem 3.5, we have

$$\begin{aligned}
&\int_1^t \left[\frac{\lambda_0}{(n-2)!}\delta(\tau(s))Q(s)\sigma^{n-2}(s) - \frac{1+p_0/\tau_0}{4}\frac{(\tau'(s))^2}{r(s)\delta(\tau(s))}\right] \mathrm{d}s \\
= &\int_1^t \left[\frac{n-1}{\theta-1}\lambda_0\alpha^{1-\theta}\beta^{n-2}s^{-1} - \frac{(p_0+\alpha)(\theta-1)}{4}\alpha^\theta s^{-1}\right] \mathrm{d}s \to \infty, \quad \text{as} \quad t \to \infty,
\end{aligned}$$

when $(n-1)\lambda_0\alpha^{1-2\theta}\beta^{n-2} > (p_0+\alpha)(\theta-1)^2/4$. This guarantees that every solution of (4.4) is oscillatory.

On the other hand, if $0 < \beta \leq \alpha \leq 1$, then $Q(t) = q(\tau(t)) = (n-1)!(\alpha t)^{\theta-n}$. When

$$\ln\frac{\alpha}{\beta} > p_0 + \alpha,$$

it follows that (3.14) or (3.15) holds, respectively. Furthermore, from Theorem 3.4, we get

$$\begin{aligned}
&\int_1^t \left[\frac{\lambda_0}{(n-2)!}\delta(s)Q(s)\sigma^{n-2}(s) - \frac{1+p_0/\tau_0}{4}\frac{1}{r(s)\delta(s)}\right] \mathrm{d}s \\
= &\int_1^t \left[\frac{n-1}{\theta-1}\lambda_0\alpha^{\theta-n}\beta^{n-2}s^{-1} - \frac{(p_0+\alpha)(\theta-1)}{4\alpha}s^{-1}\right] \mathrm{d}s \\
\geq &\int_1^t \left[\frac{n-1}{\theta-1}\lambda_0\beta^{\theta-2} - \frac{(p_0+\alpha)(\theta-1)}{4\beta}\right] s^{-1}\mathrm{d}s \to \infty, \quad \text{as} \quad t \to \infty,
\end{aligned}$$

when $(n-1)\lambda_0\beta^{\theta-1} > (p_0+\alpha)(\theta-1)^2/4$. Hence, every solution of (4.4) is oscillatory.

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